Rechenregeln für den Nabla-Operator

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\vec{\nabla}\varphi \equiv \operatorname{grad}\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$$

$$\vec{\nabla}\cdot\vec{E} \equiv \operatorname{div}\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\vec{\nabla}\times\vec{B} \equiv \operatorname{rot}\vec{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}, \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}, \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\left.\begin{aligned}\vec{\nabla}\cdot(\vec{A}\varphi) &= \varphi\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\varphi\\ \operatorname{div}(\vec{A}\varphi) &= \varphi\operatorname{div}\vec{A} + \vec{A}\cdot\operatorname{grad}\varphi\end{aligned}\right\}$$

$$\left.\begin{aligned}\vec{\nabla}\times(\vec{A}\varphi) &= \varphi\vec{\nabla}\times\vec{A} - \vec{A}\times\vec{\nabla}\varphi\\ \operatorname{rot}(\vec{A}\varphi) &= \varphi\operatorname{rot}\vec{A} - \vec{A}\times\operatorname{grad}\varphi\end{aligned}\right\}$$

$$\left.\begin{aligned}\vec{\nabla}\cdot(\vec{A}\times\vec{B}) &= \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})\\ \operatorname{div}(\vec{A}\times\vec{B}) &= \vec{B}\cdot\operatorname{rot}\vec{A} - \vec{A}\cdot\operatorname{rot}\vec{B}\end{aligned}\right\}$$

$$\left.\begin{aligned}\vec{\nabla}\times(\vec{A}\times\vec{B}) &= (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})\\ \operatorname{rot}(\vec{A}\times\vec{B}) &= (\vec{B}\cdot\operatorname{grad})\vec{A} - (\vec{A}\cdot\operatorname{grad})\vec{B} + \vec{A}\operatorname{div}\vec{B} - \vec{B}\operatorname{div}\vec{A}\end{aligned}\right\}$$

$$\left.\begin{aligned}\vec{\nabla}(\vec{A}\cdot\vec{B}) &= (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A})\\ \operatorname{grad}(\vec{A}\cdot\vec{B}) &= (\vec{B}\cdot\operatorname{grad})\vec{A} + (\vec{A}\cdot\operatorname{grad})\vec{B} + \vec{A}\times\operatorname{rot}\vec{B} + \vec{B}\times\operatorname{rot}\vec{A}\end{aligned}\right\}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\varphi) \equiv \operatorname{div}(\operatorname{grad}\varphi) \equiv \Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}, \quad \Delta = \text{Laplace-Operator}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv \operatorname{div}(\operatorname{rot}\vec{A}) = (\vec{\nabla}\times\vec{\nabla})\cdot\vec{A} \equiv 0$$

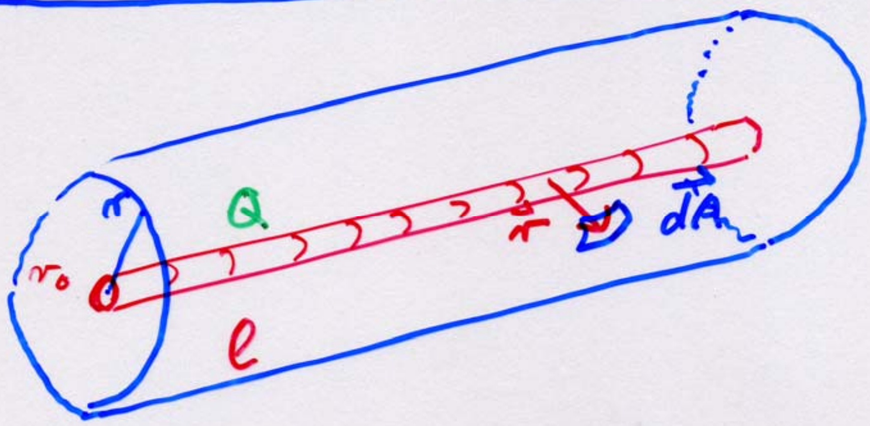
$$\vec{\nabla}\times(\vec{\nabla}\varphi) \equiv \operatorname{rot}\operatorname{grad}\varphi = (\vec{\nabla}\times\vec{\nabla})\varphi \equiv 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) \equiv \operatorname{rot}\operatorname{rot}\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{A} \equiv \operatorname{grad}\operatorname{div}\vec{A} - \Delta\vec{A}$$

Gauß: $$\int_{\text{Oberfläche}} \vec{E}\cdot d\vec{f} = \int_{\text{Volumen}} \operatorname{div}\vec{E}\, dv$$

Stokes: $$\oint_{\text{Weg}} \vec{E}\cdot d\vec{s} = \int_{\text{Fläche}} \operatorname{rot}\vec{E}\cdot d\vec{f}$$

c] Potential, Feld eines geladenen Drahtes



$\ell \gg r$

$\lambda = \frac{Q}{\ell}$

Gauß: $\frac{Q}{\varepsilon_0} = \oint_O \vec{E}\, d\vec{A}$

$\approx 2\pi r \cdot \ell \cdot E(r)$

$(\vec{E}\, d\vec{A} = |E|\,|dA|)$

$\Rightarrow$ Feldstärke: $E(r) = \frac{Q}{2\pi\varepsilon_0} \cdot \frac{1}{r}$

$\Rightarrow$ Potential: $V(r) = \int_r^\infty E(r')\,dr' = \infty$ (!)

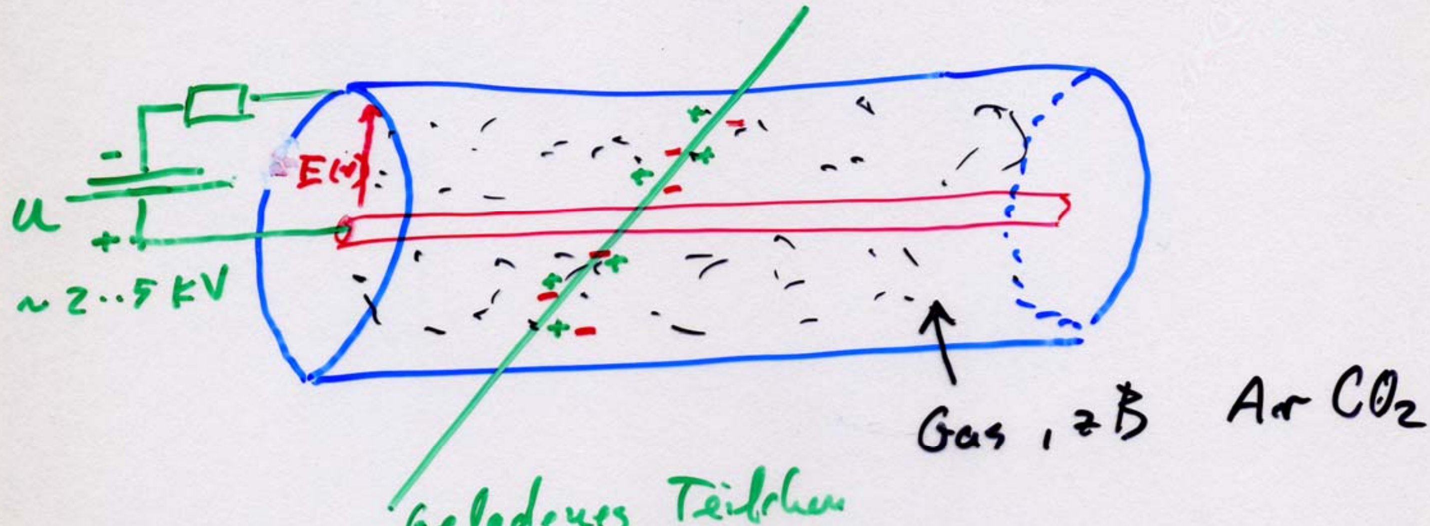
Nehme: $U(r) \equiv V(r_0) - V(r)$

$= \int_{r_0}^{\infty} \frac{\lambda}{2\varepsilon_0\pi} \cdot \frac{1}{r'}\, dr'$

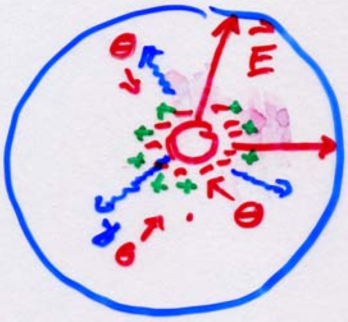
$- \int_{r}^{\infty} \frac{\lambda}{2\varepsilon_0\pi} \frac{1}{r'}\, dr'$

$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r}{r_0}$

Anwendung: Geiger-Müller-Zähler



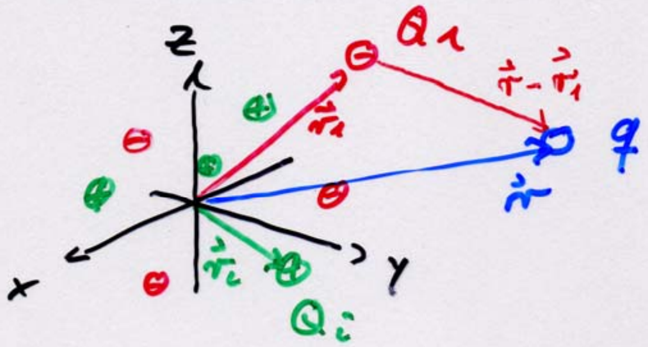
Beschleunigung der Elektronen zum Draht, der pos. Ionen zur Hülle



- Ladungslawine beim Draht
- UV Photonen von der Ionisation ionisieren erneut

$\Rightarrow$ Gasentladung

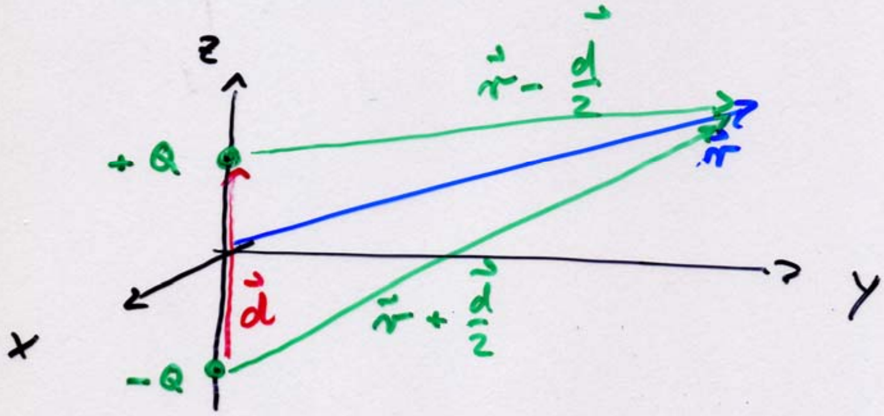
2.1.7 Multipole



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i^N \frac{Q_i}{|\vec{r}-\vec{r}_i|}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i^N \frac{Q_i}{|\vec{r}-\vec{r}_i|^2} \cdot \vec{e}_{\vec{r}-\vec{r}_i}$$

Spezialfall: Elektrischer Dipol



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\vec{r}-\frac{\vec{d}}{2}|} - \frac{Q}{|\vec{r}+\frac{\vec{d}}{2}|}\right)$$

für $|\vec{r}| \gg |\vec{d}|$: $\quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{r}\cdot\vec{d}}{r^3}Q \sim \frac{1}{r^2}$

Herleitung (Taylor)

$$\frac{1}{\left|\vec{r}+\frac{\vec{d}}{2}\right|} = \frac{1}{r\sqrt{1+\frac{\vec{r}\cdot\vec{d}}{r^2}+\underbrace{\frac{\vec{d}^2}{4r^2}}_{\text{vernachlässigt}}}}$$

$$\approx \frac{1}{r}\cdot\left(1-\frac{\vec{r}\vec{d}}{2r^2}\right)$$

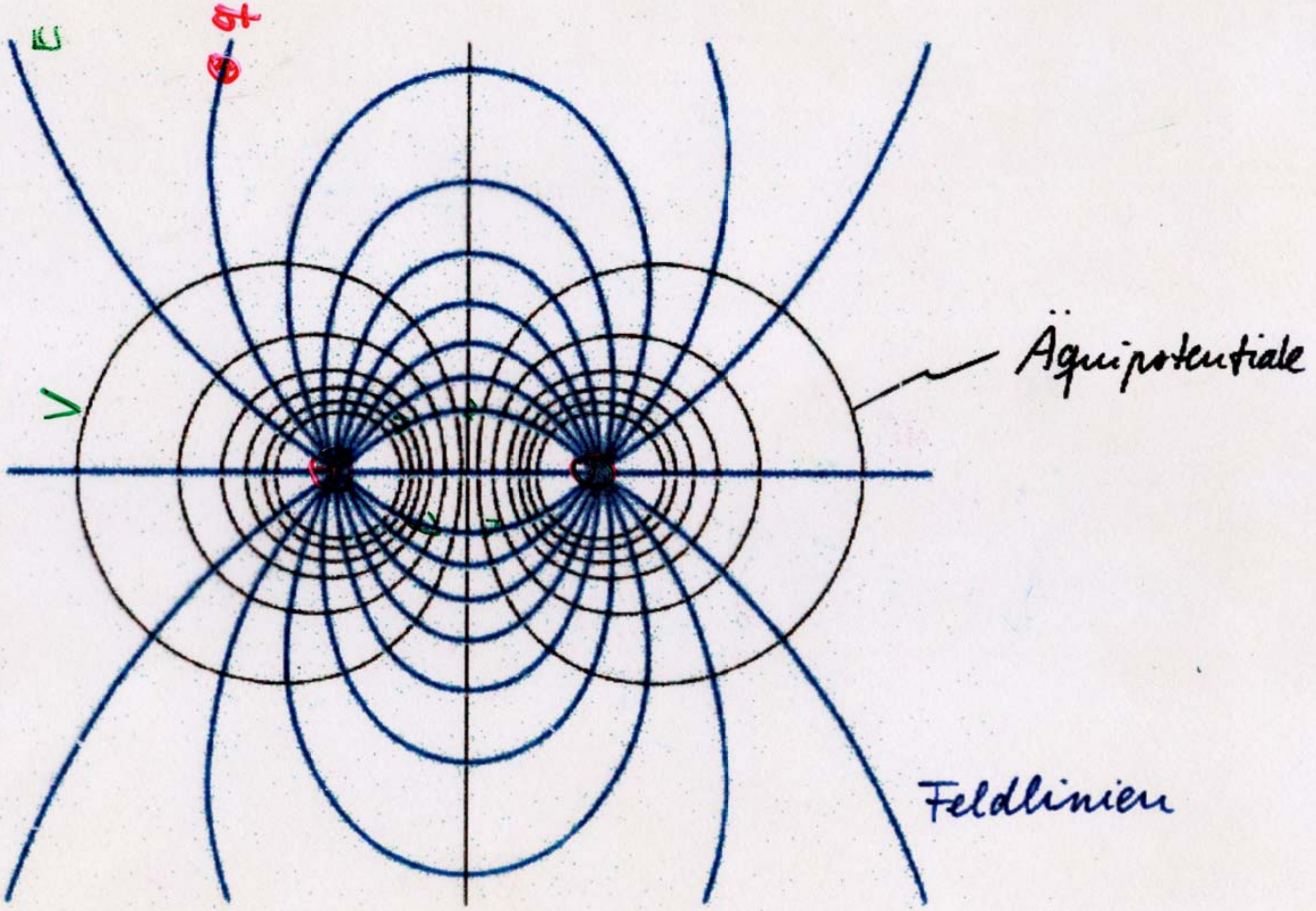
$$(1+\varepsilon)^{-\frac{1}{2}} \approx 1-\frac{\varepsilon}{2}$$

$$\frac{1}{\left|\vec{r}-\frac{\vec{d}}{2}\right|} \approx \frac{1}{r}\left(1+\frac{\vec{r}\vec{d}}{2r^2}\right)$$

Dipolmoment: $\vec{p} = Q\cdot\vec{d}$

$$V(\vec{r}) = \frac{p\cdot\cos\theta}{4\pi\varepsilon_0}\ \frac{1}{r^2}$$

Dipol



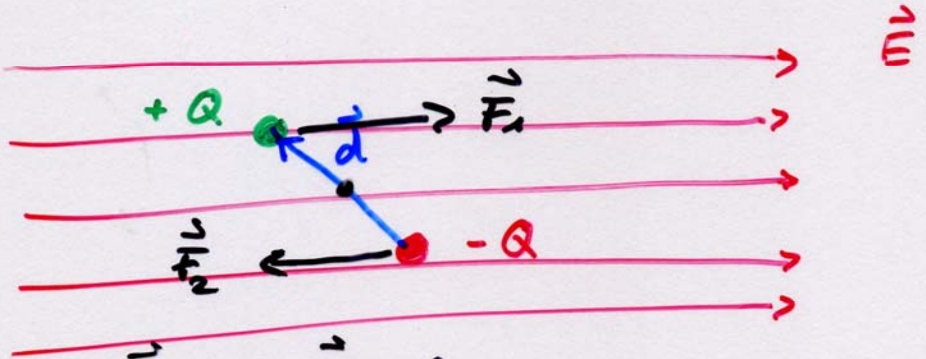
$$\vec{E}(\vec{r}) = - \nabla V(\vec{r})$$

$$= - \frac{Q}{4\pi\varepsilon_0} \nabla \cdot \left(\frac{\vec{d}\,\vec{r}}{r^3} \right)$$

$$= - \frac{Q}{4\pi\varepsilon_0} \left(\frac{\vec{d}}{r^3} - 3 \cdot \frac{\vec{d}\,\vec{r}}{r^4} \vec{e}_r \right)$$

Kräfte auf einen Dipol

i) Im homogenen $\vec{E}$ - Feld



$$\vec{F}_1 = \vec{E} \cdot Q$$
$$\vec{F}_2 = - \vec{E} \cdot Q$$

=> Dipol erfährt ein Drehmoment, wird ausgerichtet $\parallel \vec{E}$

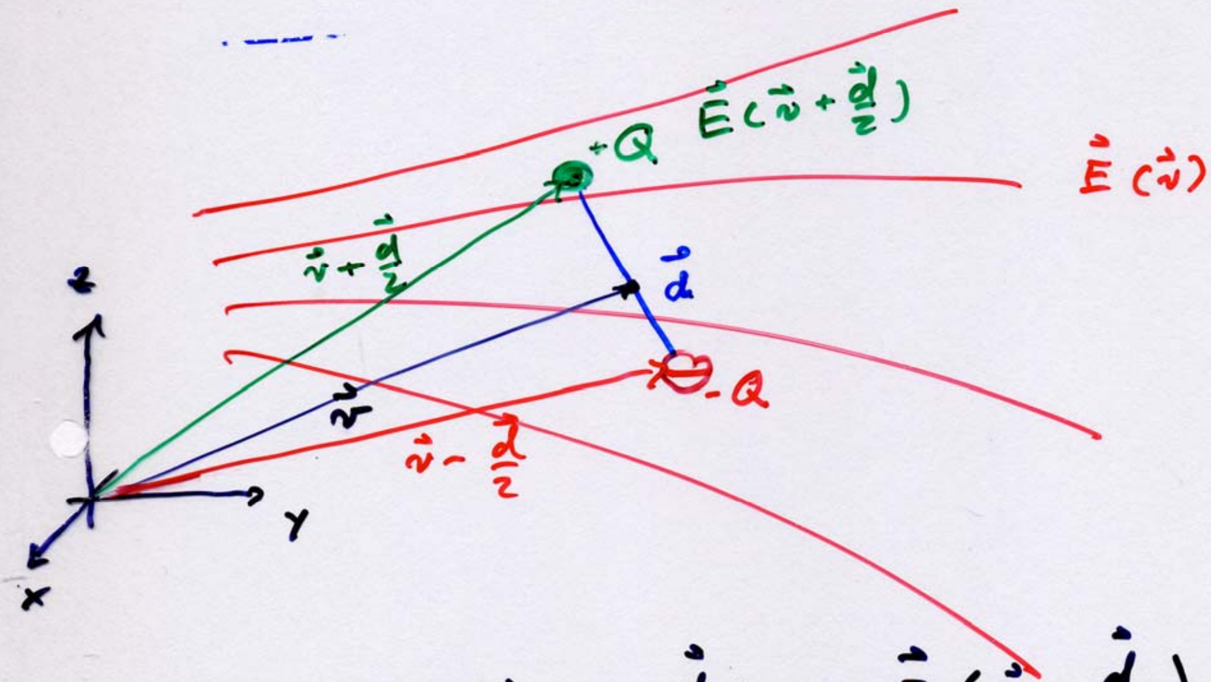
$$\vec D = \vec r \times \vec F$$

$$= \frac{\vec d}{2} \times \vec F_1 + \frac{-\vec d}{2} \times \vec F_2$$

$$= \frac{\vec d \times \vec E}{2} Q + \frac{\vec d \times \vec E}{2} \cdot Q = \vec d \times \vec E \cdot Q$$

$=0$, wenn $\vec d \parallel \vec E$

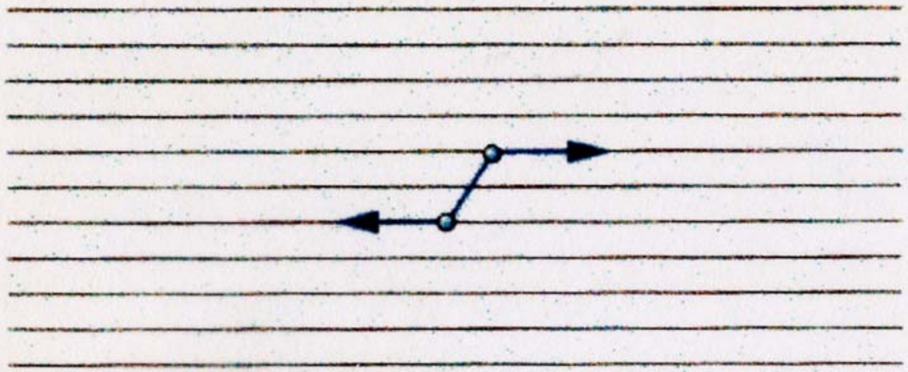
ii) Im inhomogenen $\vec E$-Feld:



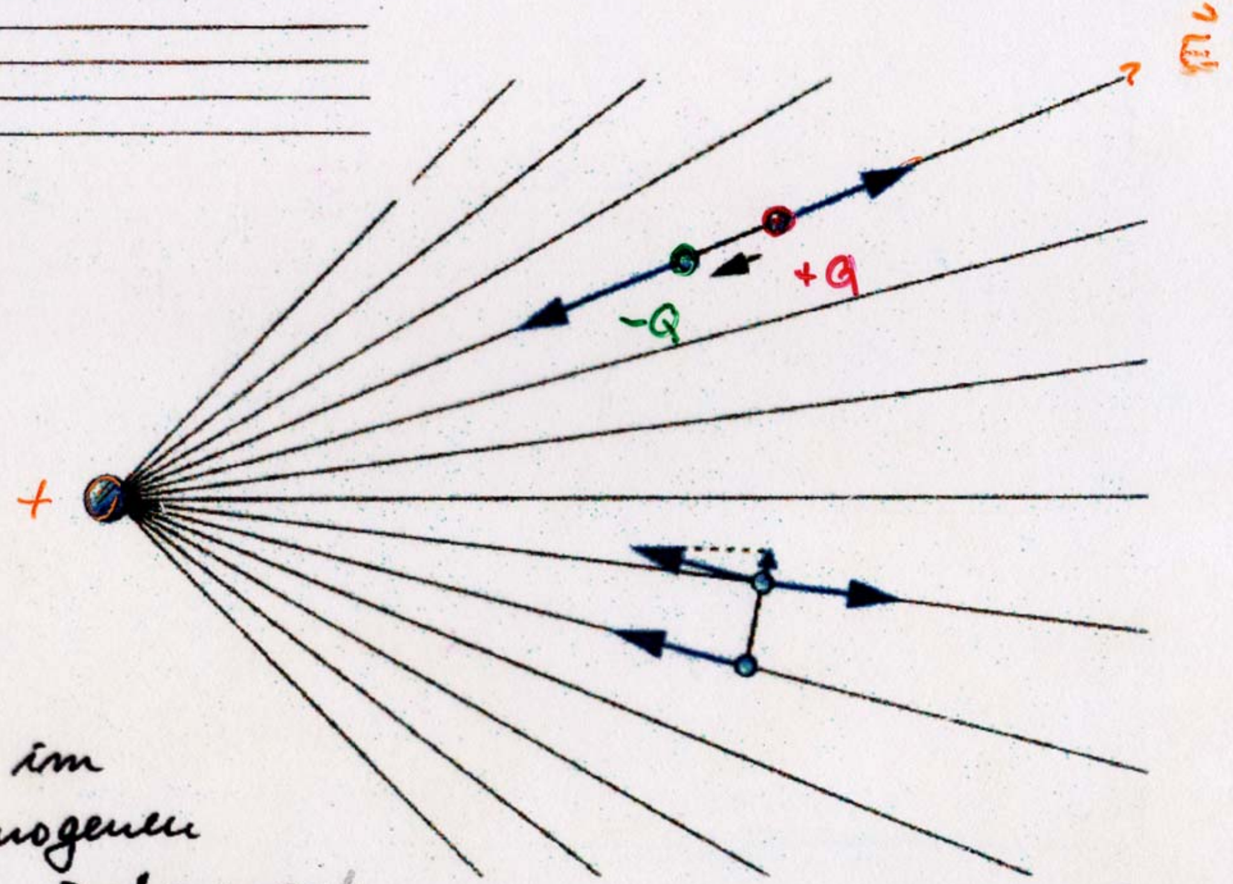
$$\vec F = Q\left(\vec E\left(\vec r + \frac{\vec d}{2}\right) - \vec E\left(\vec r - \frac{\vec d}{2}\right)\right)$$

$$\approx Q \cdot \left(\vec E(\vec r) + \left(\frac{\vec d}{2}\vec\nabla\right)\vec E(\vec r) - \left(\vec E(\vec r) - \left(\frac{\vec d}{2}\nabla\right)\vec E(\vec r)\right)\right)$$

$$\approx Q \cdot (\vec d\, \vec\nabla)\, \vec E = (\vec p\, \vec\nabla)\, \vec E \neq 0$$



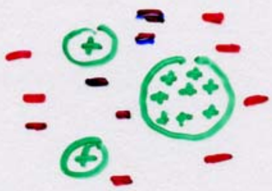
Dipol im homogenen
Feld: nur Drehmoment



Dipol im
inhomogenen
Feld: Drehmoment

Dipole in der Natur :

z.B Wasser : H_2O



$\vec{p}$

Dipolmoment $p_{H_2O} = 6 \cdot 10^{-30}$ C m

Auch : Neutron n

$p_n < 2{,}9 \cdot 10^{-26}$ e cm

$< 5 \cdot 10^{-47}$ C m

2.1.8 Divergenz des Elektrischen Feldes

a) Aus $U = \Delta V = -\int_{\vec{r}_1,\,P}^{\vec{r}_2} \vec{E}\, d\vec{s}$

$$= -\int_{\vec{r}_1}^{\vec{r}_2} (E_x\, dx + E_y\, dy + E_z\, dz)$$

für $x_1 \to x_2$, $y_1 \to y_2$, $z_1 \to z_2$: $\Delta V_x = -E_x\, dx$

$$\Rightarrow E_x = -\frac{\partial V}{\partial x}$$

$$\Delta V_y = -E_y\, dy$$

$$\Rightarrow E_y = -\frac{\partial V}{\partial y}$$

ΔV_z ...

$$\Rightarrow E_z = -\frac{\partial V}{\partial z}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$